3. Derived Categories of Coherent Sheaves

〈あらすじ〉 X : smooth, projective. $D^b(X) := D^b(\mathrm{Coh}(X))$

1. Basic structure

- (3.10) $D^b(X)$ が indecomposable $\iff$ X : 連結.
- (3.12, Serre Duality) $D^b(X)$ は Serre functor を持つ.

2. Spanning classes in the derived category

- (3.17) $(k(x))_{x \in X}$ は $D^b(X)$ の Spanning class.
- (3.19) L : ample $(L^i)_{i \in \mathbb{Z}}$ は $D^b(X)$ の Spanning class.

3. Derived functors in algebraic geometry

- 各種消滅定理と有限性定理, 関手ごとの acyc. obj を使い

$$\left\{\begin{array}{ll} \Gamma, f_*, \mathcal{H}om & \text{の右導来} \\ \mathrm{tr}, (-)^\vee & \text{の延長} \\ \otimes, f^* & \text{の左導来} \end{array}\right. \quad \text{を } D^b(\mathrm{Coh}) \text{ の中におさめる.}$$

- それらの交換 (proj. formula, $Lf^* \dashv Rf_*$, flat base change など) をみる.

4. Grothendieck-Verdier duality

- ある意味での Rf_* の右随伴 $f^!$ を構成する.

導来圏で初めて定義できる関手.

1. Basic structure　　X : Noether.

$D^*(X) := D^*(\mathrm{Coh}(X))$: 連接層の導来圏

$$\begin{tikzcd}
\mathrm{Sh}_{\mathcal{O}_X}(X) \arrow[r, dotted] & D^*(\mathrm{Sh}_{\mathcal{O}_X}(X)) & \supseteq & D^*_{qcoh}(\mathrm{Sh}_{\mathcal{O}_X}(X)) \\
\cup & \cup & & (\text{Cor. } 3.4) \\
\mathrm{Qcoh}(X) \arrow[r, dotted] & D^*(\mathrm{Qcoh}(X)) & \supseteq & D^*_{coh}(\mathrm{Qcoh}(X)) \arrow[uu, phantom] \\
\cup & \cup & {*=b\text{で}} & (\text{Prop. } 3.5) \\
\mathrm{Coh}(X) \arrow[r, dotted] & D^*(X) \arrow[uurr, "\sim"]
\end{tikzcd}$$

Prop. 2.4.2.
$+ \begin{cases} \circ\ \mathrm{Qcoh}(X) \text{ は enough inj.} \\ \circ\ \text{―――― は } \mathrm{Sh}_{\mathcal{O}_X}(X) \text{ の thick sub.} \end{cases}$

$\mathcal{G}^\bullet \in D^b_{coh}(\mathrm{Qcoh}(X))$.

(3.6): $\underset{\mathrm{Qcoh}}{\mathcal{G}} \twoheadrightarrow \underset{\mathrm{coh}}{\mathcal{F}}$ なる $\exists \underset{\mathrm{coh.}}{\mathcal{G}'} \hookrightarrow \mathcal{G}$, $\mathcal{G}' \to \mathcal{F}$ ここも全射.

$$\begin{tikzcd}[column sep=small]
 & & & & \text{coh.}: \mathcal{G}^j_2 \ (3) \arrow[d, "3.6"] \arrow[drr] & & & \\
 & & & \mathrm{Im}\, d^{j-1} \arrow[r, hook] & \mathrm{Ker}\, d^j \arrow[r, two heads] \arrow[d] & & \mathcal{H}^j : \text{coh.} & \\
 & \mathcal{G}^{j-1}_{new}\ (5) \arrow[rr, "d^{j-1}_{new}"] \arrow[dr] & & \mathcal{G}^j_{new}\ (4) \arrow[dr] \arrow[rrr, "d^j_{new}"] & & & & \text{coh.} \\
\cdots \arrow[r] & \mathcal{G}^{j-2} \arrow[u, "d^{j-2}_{new}"] \arrow[r, "d^{j-2}"'] & \mathcal{G}^{j-1} \arrow[uur] \arrow[ur, "\text{p.b.}"] \arrow[rr, "d^{j-1}"'] & & \mathcal{G}^j \arrow[rr, "d^j"'] \arrow[dr] & & \mathcal{G}^{j+1} \arrow[r, dash] & \cdots \\
 & & & \text{coh.}: \mathcal{G}^j_1\ (2) \arrow[ur, "3.6"] \arrow[rr, two heads] & & \mathrm{Im}\, d^j\ (1)\,\text{coh.} \arrow[r, hook] & \mathrm{Ker}\, d^{j+1} \arrow[u] \arrow[r, two heads] & \mathcal{H}^{j+1} : \text{coh.}
\end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
\cdots \arrow[r] & \mathcal{G}^{j-2} \arrow[rr, "d^{j-2}_{new}"] \arrow[dd, equal] \arrow[dr] & & \mathcal{G}^{j-1}_{new} \arrow[rr, "d^{j-1}_{new}"] \arrow[dd] \arrow[rrdd, phantom, "\text{p.b.}"] & & \mathcal{G}^j_{new} \arrow[rr, "d^j_{new}"] \arrow[dd] \arrow[dr] & & \mathcal{G}^{j+1} \arrow[r, dash] \arrow[dd, equal] & \cdots & = \mathcal{G}^\bullet_{new} \arrow[dd, "\wr\ \mathrm{qis.}"] \\
 & & \begin{smallmatrix} \mathrm{Im} \hookrightarrow \mathrm{Ker} \\ \parallel\ (7)\ \parallel \\ \mathrm{Im} \hookrightarrow \mathrm{Ker} \end{smallmatrix} \arrow[ur] & & & & \begin{smallmatrix} \mathrm{Im} \hookrightarrow \mathrm{Ker} \\ \parallel\ (6)\ \parallel \\ \mathrm{Im} \hookrightarrow \mathrm{Ker} \end{smallmatrix} \arrow[ur] \arrow[dr] & & & \\
\cdots \arrow[r] & \mathcal{G}^{j-2} \arrow[rr, "d^{j-2}"'] \arrow[ur] & & \mathcal{G}^{j-1} \arrow[rr, "d^{j-1}"'] & & \mathcal{G}^j \arrow[rr, "d^j"'] \arrow[ur] & & \mathcal{G}^{j+1} \arrow[r, dash] & \cdots & = \mathcal{G}^\bullet
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Ker}\, d^{j-1}_{new} \arrow[r, hook] \arrow[d, equal] & \mathcal{G}^{j-1}_{new} \arrow[rr, "d^{j-1}_{new}"] \arrow[d] \arrow[dr, two heads] & & \mathcal{G}^j_{new} \arrow[r, two heads] \arrow[d] & \mathrm{Coker}\, d^{j-1}_{new} \arrow[d] \\
 & {} \arrow[r, phantom, "\text{p.b.}"] & \mathrm{Im} \arrow[ur, hook] \arrow[d] & {} & {} \arrow[r, phantom, "\text{これは bij. } (9)"] & {} \\
\mathrm{Ker}\, d^{j-1} \arrow[r, hook] & \mathcal{G}^{j-1} \arrow[rr, "d^{j-1}"'] \arrow[r, two heads] \arrow[d] & \mathrm{Im} \arrow[r, hook] & \mathcal{G}^j \arrow[r, two heads] \arrow[d] & \mathrm{Coker}\, d^{j-1} \\
0 \arrow[r, hook] & \mathcal{E}^{j-1} \arrow[rr] & & \mathcal{E}^j \arrow[r, two heads] & 0 \quad : \text{ex. } (10)
\end{tikzcd}$$

□

$\mathrm{Supp}\,\mathcal{F}^\bullet := \bigcup \mathrm{Supp}\,\mathcal{H}^i(\mathcal{F}^\bullet)$: $\mathcal{F}^\bullet \in D^b(X)$ の support

Lem.3.9.

$\mathrm{Supp}\,\mathcal{F}^\bullet = Z_1 \sqcup Z_2$ なら,

$\mathcal{F}^\bullet = \mathcal{F}_1^\bullet \oplus \mathcal{F}_2^\bullet \quad (\mathrm{Supp}\,\mathcal{F}_i^\bullet \subseteq Z_i)$ とかける.

prf.

$\mathcal{H}^m(\mathcal{F}^\bullet) \neq 0$. 層として $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \ (\mathrm{Supp}\,\mathcal{H}_i \subseteq Z_i)$

$$\begin{array}{ccccccccccccc}
 & & & & \mathcal{H}^m(\mathcal{F}^\bullet) \neq 0 & & & & & & & & \\
 & & & & \| & & & & & & & & \\
[\ \cdots & \to & 0 & \to & \mathcal{H} & \to & 0 & \to & \cdots\ ] & = & \mathcal{H}[-m] & & \\
 & & & & \downarrow & & & & & & \downarrow & & \\
[\ \cdots & \to & 0 & \to & \mathcal{F}^m & \to & \mathcal{F}^{m+1} & \to & \cdots\ ] & = & \mathcal{F}^\bullet & & \\
 & & & & \downarrow & & \downarrow & & & & \downarrow & & \\
[\ \cdots & \to & 0 & \to & 0 & \to & \mathcal{G}^{m+1} & \to & \cdots\ ] & = & \mathcal{G}^\bullet & = & \mathcal{G}_1^\bullet \oplus \mathcal{G}_2^\bullet \ (\mathrm{Supp}\,\mathcal{G}_i^\bullet \subseteq Z_i) \\
 & & & & & & & & & & \downarrow{\scriptstyle +1} & & \\
 & & & & & & & & & & \mathcal{H}[1-m] & & :\mathrm{d.t.}
\end{array}$$

コホモロジーは一致 ⇝ 帰納法の仮定

(コホモロジーのsuppまでしか分からない)

$$E_2^{p,q} = \mathrm{Hom}(\underbrace{\mathcal{H}^{-q}(\mathcal{G}_1^\bullet)}_{\mathrm{Supp} \subseteq Z_1}, \underbrace{\mathcal{H}_2[p]}_{\mathrm{Supp} \subseteq Z_2}) \Rightarrow \mathrm{Hom}(\mathcal{G}_1^\bullet, \mathcal{H}_2[p+q])$$

$= 0$ ※少し非自明 $\Longrightarrow$ $= 0$

特に $\mathrm{Hom}(\mathcal{G}_1^\bullet, \mathcal{H}_2[1-m]) = 0$

同様に $\qquad 2 \qquad 1 \qquad = 0$

$$\rightsquigarrow \mathrm{Hom}(\mathcal{G}^\bullet, \mathcal{H}[1-m]) = \bigoplus_{i=1,2} \mathrm{Hom}(\mathcal{G}_i^\bullet, \mathcal{H}_i[1-m])$$

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow & & \downarrow \\
\exists\,\mathcal{F}_i^\bullet & & \mathrm{Hom}(\mathcal{F}_i^\bullet, A) & & \mathrm{Hom}(\oplus\mathcal{F}_i^\bullet, A) & & \mathrm{Hom}(\mathcal{F}^\bullet, A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{G}_i^\bullet & & \mathrm{Hom}(\mathcal{G}_i^\bullet, A) & & \mathrm{Hom}(\oplus\mathcal{G}_i^\bullet, A) & = & \mathrm{Hom}(\mathcal{G}^\bullet, A) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{H}_i[1-m] & & \mathrm{Hom}(\mathcal{H}_i[1-m], A) & & \mathrm{Hom}(\oplus\mathcal{H}_i[1-m], A) & = & \mathrm{Hom}(\mathcal{H}[1-m], A) \\
{\scriptstyle +1}\downarrow\ :\mathrm{d.t.} & & \downarrow\ :\mathrm{ex.} & & \downarrow\ :\mathrm{ex.} & & \downarrow\ :\mathrm{ex.} \\
 & & \vdots & & \vdots & & \vdots
\end{array}$$

: ここに iso が誘導される.

$\mathrm{Supp}\,\mathcal{F}_i \subseteq Z_i$.

□

Prop. 3.10. X : Noether.

$D^b(X)$ が indecomposable

$\Longleftrightarrow X$: 連結

Definition 1.52

A triangulated category $\mathscr{D}$ is decomposed into triangulated subcategories $\mathscr{D}_1, \mathscr{D}_2 \subset \mathscr{D}$ *if the following three conditions are satisfied:*

i) *Both categories $\mathscr{D}_1$ and $\mathscr{D}_2$ contain objects non-isomorphic to 0.*

ii) *For all $A \in \mathscr{D}$ there exists a distinguished triangle*

$$B_1 \to A \to B_2 \to B[1]$$

with $B_i \in \mathscr{D}_i, i = 1, 2$.

iii) $\mathrm{Hom}(B_1, B_2) = \mathrm{Hom}(B_2, B_1) = 0$ *for all $B_1 \in \mathscr{D}_1$ and $B_2 \in \mathscr{D}_2$.*

A triangulated category that cannot be decomposed is called indecomposable.

prf. $\Rightarrow$ の対偶はさっきの Lemma.

$\Leftarrow$: 連結なのに $\mathcal{D}_1, \mathcal{D}_2 \subseteq D^b(X)$ で分解されたとする.

① $\mathcal{O}_X \in D^b(X)$ を考える.

$\mathcal{O}_X = \underset{\mathcal{I}_{X_1}}{\underset{\shortparallel}{\mathcal{F}_1}} \oplus \underset{\mathcal{I}_{X_2}}{\underset{\shortparallel}{\mathcal{F}_2}}$ ($\mathcal{F}_i \in \mathcal{D}_i$) なら、各々イデアル層なので閉部分スキーム X_i を定義する.

なる $X_1 \cap X_2 = \emptyset$ かつ $X_1 \cup X_2 = X$ で、連結なので

X_1 or $X_2 = X$ i.e. $\mathcal{F}_1$ or $\mathcal{F}_2 = 0$. つまり $\mathcal{O}_X$ は $\mathcal{D}$ のどちらかに入る.

$\mathcal{O}_X \in \mathcal{D}_1$ だとする.

② $k(x) \in D^b(X)$ を考える ($x \in X$)

$k(x)$ は単純なので分解は自明で、$\mathcal{D}_1$ か $\mathcal{D}_2$ のどちらかに入る.

非自明な $\mathcal{O}_X \longrightarrow k(x)$ があるので $k(x) \in \mathcal{D}_1$.

③ $0 \neq \mathcal{F}^\bullet \in \mathcal{D}_2$ が存在したと仮定する.

$\mathcal{H}^m(\mathcal{F}^\bullet) =: \mathcal{H}^m \neq 0$ $\quad x \in \mathrm{Supp}\,\mathcal{H}^m$ をとると、非自明な $\mathcal{H}^m \twoheadrightarrow k(x)$ がある.

$$\begin{array}{ccccccccccc}
[\cdots \to & \mathcal{F}^{m-1} & \to & \mathcal{F}^m & \to & 0 & \to \cdots] & = & \mathcal{F}^\bullet & \in & \mathcal{D}_2 \\
 & & & \downarrow & & & & & \downarrow & & \\
[\cdots \to & 0 & \to & \mathcal{H}^m & \to & 0 & \to \cdots] & = & \mathcal{H}^m[-m] & & \\
 & & & \downarrow & & & & & \downarrow & & \\
[\cdots \to & 0 & \to & k(x) & \to & 0 & \to \cdots] & = & k(x)[-m] & \in & \mathcal{D}_1
\end{array}$$

$\therefore$ 合成は非自明で iii) に矛盾. $\square$

以降ずっと X : smooth, proj. var / k,　$\dim = n$.

$$S_X : D^b(X) \xrightarrow{\sim} D^b(X) \; ; \; \mathcal{F} \mapsto \mathcal{F} \otimes \omega_X [n] \quad : \text{exact}.$$

Thm. 3.12. (Serre duality)

S_X は $D^b(X)$ の Serre functor である.

すなわち、$\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(X)$ に対し関手的な同型

$$\eta_{\mathcal{E}^\bullet, \mathcal{F}^\bullet} : \mathrm{Hom}_{D^b(X)}(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{D^b(X)}(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X[n])^\vee$$

または　$\mathrm{Ext}^i_X(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \xrightarrow{\sim} \mathrm{Ext}^{n-i}_X(\mathcal{F}^\bullet, \mathcal{E}^\bullet \otimes \omega_X)^\vee$　を主張する.

Prop. 3.13.

$\mathrm{cd}(\mathrm{Coh}(X)) = n$.　i.e.　Ext^i は $i > n$ で消える.
$i = n$ で消えないことがある.

Cor. 3.14.

$\mathrm{RHom}^\bullet(\mathcal{E}^\bullet, \mathcal{F}^\bullet) \in D^b(Ab)$　$(\mathcal{E}^\bullet, \mathcal{F}^\bullet \in D^b(X))$

Cor. 3.15 C : sm. proj. curve.

$D^b(C)$ の対象は $\bigoplus \mathcal{E}_i[i]$ $\left(\mathcal{E}_i \in \mathrm{Coh}(X)\right)$ の形と同型.

prf

$$\begin{tikzcd}
{[\cdots \to 0 \to \mathcal{H}^{i_0}(\mathcal{E}^\bullet) \to 0 \to \cdots]} \arrow[r, phantom, "="] & \mathcal{H}^{i_0}(\mathcal{E}^\bullet)[-i_0] \arrow[d] \\
{[\cdots \to 0 \to \mathcal{E}^{i_0} \to \mathcal{E}^{i_0+1} \to \cdots]} \arrow[r, phantom, "="] & \mathcal{E}^\bullet \arrow[d] \\
{[\cdots \to 0 \to 0 \to \mathcal{E}_1^{i_0+1} \to \cdots]} & \mathcal{E}_1^\bullet = \bigoplus_{i>i_0} \mathcal{H}^i(\mathcal{E}_1^\bullet)[-i] \arrow[d] \\
 & \mathcal{H}^{i_0}(\mathcal{E}^\bullet)[1-i_0] \arrow[u, bend right, "=0\ \text{split}"']
\end{tikzcd}$$

コホモロジーは一致　　帰納法の仮定

$$\mathrm{Hom}\left(\mathcal{E}_1^\bullet, \mathcal{H}^{i_0}(\mathcal{E}^\bullet)[1-i_0]\right) \cong \bigoplus_{i>i_0} \mathrm{Ext}^{\underbrace{1+i-i_0}_{\geq 2}}\left(\mathcal{H}^i(\mathcal{E}_1^\bullet), \mathcal{H}^{i_0}(\mathcal{E}^\bullet)\right) = 0$$

$\because \mathrm{cd}(C) = 1$ $\square$

2. Spanning classes in the derived category　　X : sm. proj. var.

Prop. 3.17.
$(k(x))_{x : X の閉点}$ は $D^b(X)$ の spanning class.

Definition 1.47

A collection Ω of objects in a triangulated category $\mathscr{D}$ is a spanning class of $\mathscr{D}$ (or spans $\mathscr{D}$) if for all $B \in \mathscr{D}$ the following two conditions hold:

i) *If* $\mathrm{Hom}(A, B[i]) = 0$ *for all* $A \in \Omega$ *and all* $i \in \mathbb{Z}$, *then* $B \simeq 0$.

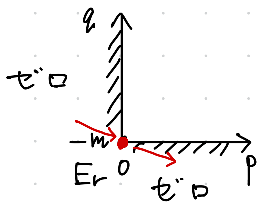
ii) *If* $\mathrm{Hom}(B[i], A) = 0$ *for all* $A \in \Omega$ *and all* $i \in \mathbb{Z}$, *then* $B \simeq 0$.

Serre functor があれば i) ii) は片方でよい (Ex. 1.48)

prf.　$0 \neq \mathcal{F}^\bullet \in D^b(X)$ を取る. $\mathcal{H}^q := \mathcal{H}^q(\mathcal{F}^\bullet)$ として

$E_2^{p,q} = \mathrm{Hom}(\mathcal{H}^{-q}, k(x)[p]) \Rightarrow E^{p+q} = \mathrm{Hom}(\mathcal{F}^\bullet, k(x)[p+q])$ がある.

$\mathcal{H}^m \neq 0$ なる最大の m を取り, $x \in \mathrm{Supp}\,\mathcal{H}^m$ を取る (i.e. $\mathcal{H}^m \xrightarrow[\neq 0]{} k(x)$ がある)



$p < 0$ か $q < -m$ で $E_r^{p,q} = 0$ なので

$E_2^{0,-m} = E_3^{0,-m} = \cdots = E_\infty^{0,-m}$.　よって

$\mathrm{Hom}(\mathcal{F}^\bullet, k(x)[-m]) = E^{-m} = E_\infty^{0,-m} = \mathrm{Hom}(\mathcal{H}^m, k(x)) \neq 0$

□.

Prop. 3.18　X : proj. var. $/k$ (つまり特異でもよい)
L : ample なら $(L^i)_{i \in \mathbb{Z}}$ は $\mathrm{Coh}(X)$ の ample seq.

prf.　i) ii) は OK. iii) は smooth なら Serre duality で簡単.

一般の場合, L : very ample に帰着し, 各閉点 x ごとに

そこで消える L の大域切断 s_x がある. $0 \neq \varphi \in \mathrm{Hom}(\mathcal{F}, L^i)$ なら $\varphi|_x \neq 0$ なる

x に対し, $\mathrm{Hom}(\mathcal{F}, L^{i-1}) \xhookrightarrow{s_x} \mathrm{Hom}(\mathcal{F}, L^i)$ の像に φ が入っておらず,

各々の有限次元性から $i \ll 0$ で $\mathrm{Hom}(\mathcal{F}, L^i) = 0$ となる.　□

Cor. 3.19　X : sm. proj. var. $/k$.
L : ample なら $(L^i)_{i \in \mathbb{Z}}$ は $D^b(X)$ の spanning class.

prf.　sm. より $\mathrm{cd}(\mathrm{Coh}(X)) < \infty$ なので Prp. 2.73 を適用できる.　□

3. Derived functors in algebraic geometry

$\mathcal{A}$: enough inj.

モデルケース　　　　必要なものリスト

$\boxed{1-1}$ Abel圏の関手の右導来

$$F : \mathcal{A} \longrightarrow \mathcal{B} \ : \text{left ex.}$$
$$\Downarrow$$
$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}) \ : \text{導来}$$

(特になし)

$\boxed{1-2}$ ホモトピー圏の関手の右導来 (Rmk.2.51.)

$$F : K^+(\mathcal{A}) \longrightarrow K^+(\mathcal{B}) \ : \ \text{ex.}$$
$$\Downarrow$$
$$RF : D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B}) \ : \text{導来}$$

☆ F-adapted class が存在すること.

($\mathcal{K} \subseteq K^+(\mathcal{A})$: tri.sub が F-adapted

$\overset{\text{def}}{\Longleftrightarrow}$ $\begin{cases} \circ\ F(\mathcal{K} \cap \text{acyc.}) \subseteq \text{acyc.} \\ \circ\ K^+(\mathcal{A}) \text{ は enough } \mathcal{K}. \end{cases}$

i.e. $\forall A \in K^+(\mathcal{A}),\ \exists A \xrightarrow{\text{qis}} K \in \mathcal{K}$

$\boxed{2-1}$ 右導来の部分圏制限 (Cor.2.68(i))

$$\begin{tikzcd}
 & D^+(\mathcal{A}) \arrow[r, "RF"] & D^+(\mathcal{B}) \arrow[dr, "h^i"] & \\
\mathcal{A} \arrow[ur] & & & \mathcal{B} \\
 & D^+(\mathcal{A}') \arrow[r] \arrow[uu, hook] & D^+_{\mathcal{B}'}(\mathcal{B}) \arrow[dr] \arrow[uu, hook] & \\
\mathcal{A}' \arrow[ur] \arrow[uu, hook] \arrow[rrr, dotted, "R^iF|_{\mathcal{A}'}"] & & & \mathcal{B}' \arrow[uu, hook, "\text{thick subcat}"']
\end{tikzcd}$$

☆ $\overset{R^iF|_{\mathcal{A}'}}{\cdots\cdots\rightarrow}$ が存在すること.

(つまり, $\forall A \in \mathcal{A}', \forall i,\ R^iF(A) \in \mathcal{B}'$.

☆ $i \ll 0$ で $\cdots\cdots\rightarrow$ は 0 であること.

(大抵OK)

$\boxed{2-2}$ 右導来の有界制限 (Cor.2.68(ii))

$$\begin{tikzcd}
 & D^+(\mathcal{A}) \arrow[r, "RF"] & D^+(\mathcal{B}) \\
 & D^b(\mathcal{A}) \arrow[r] \arrow[u, hook] & D^b(\mathcal{B}) \arrow[u, hook] \\
\mathcal{A} \arrow[ur] \arrow[urr, dotted] & &
\end{tikzcd}$$

☆ $\cdots\cdots\rightarrow$ が存在すること.

(つまり, $\forall A \in \mathcal{A},\ RF(A) \in D^b(\mathcal{B})$)

$\Longleftrightarrow$ R^iF の消滅定理

$\boxed{3-1}$ Abel圏の関手の左導来

$$F : \mathcal{A} \longrightarrow \mathcal{B} \quad : \text{right ex.}$$
$$\Downarrow$$
$$LF : D^-(\mathcal{A}) \longrightarrow D^-(\mathcal{B}) \quad : \text{導来}$$

☆ F-adapted class が存在すること。

$\mathcal{P} \subseteq \mathcal{A}$: 有限和で閉 が F-adapted

$\overset{\text{def}}{\Longleftrightarrow}$
- ∘ $F(\mathcal{P}\text{複体}^- \cap \text{acyc.}) \subseteq \text{acyc.}$
- ∘ $\mathcal{A}$ は enough $\mathcal{P}$.
  i.e. $\forall A \in \mathcal{A},\ \mathcal{P} \ni {}^{\exists}P \twoheadrightarrow A$

$\boxed{3-2}$ ホモトピー圏の関手の左導来

$\boxed{4-1}$ 左導来の部分圏制限

$\boxed{4-2}$ 左導来の有界制限

右バージョンと同様。

$\boxed{5-1}$ 双関手の導来

$$\begin{array}{ccccc}
F : \mathcal{A} & \times & \mathcal{A} & \longrightarrow & \mathcal{B} \\
\downarrow & & \downarrow & & \downarrow \\
K^-(\mathcal{A}) & \times & K^-(\mathcal{A}) & \longrightarrow & K^-(\mathcal{B}) \\
\| & & & & \Downarrow \boxed{3-1}\ 1\&3 \\
K^-(\mathcal{A}) & \times & D^-(\mathcal{A}) & \longrightarrow & D^-(\mathcal{B}) \\
\downarrow & 2 & \| & & \| \\
LF : D^-(\mathcal{A}) & \times & D^-(\mathcal{A}) & \longrightarrow & D^-(\mathcal{B}) \quad : \text{導来}
\end{array}$$

F : それぞれ right ex.

☆ F-adapted class(?) が存在すること。

$\mathcal{P} \subseteq \mathcal{A}$: 有限和で閉 が F-adapted

$\overset{\text{def}}{\Longleftrightarrow}$
- ∘[1] $F(\text{任意}, \mathcal{P}\text{複体}^- \cap \text{acyc.}) \subseteq \text{acyc.}$
- ∘[2] $F(\text{acyc}, \mathcal{P}\text{複体}^-) \subseteq \text{acyc.}$
- ∘[3] $\mathcal{A}$ は enough $\mathcal{P}$.
  i.e. $\forall A \in \mathcal{A},\ \mathcal{P} \ni {}^{\exists}P \twoheadrightarrow A$

1&3 … $\forall A \in K^-(\mathcal{A})$, $\mathcal{P}$ は $F(A,-)$-adapted

2 … $F(-, \mathcal{P}\text{複体})$ は完全関手

固定データ	スタート	adapted class	$h^i \circ -$	$D^+(Qcoh)$	$D^b(Qcoh)$	$D^\pm(Coh)$	$D^b(Coh)$
$X \xrightarrow{f} k$	$\Gamma : Qcoh(X) \to Vec(k)$	$\{\text{injective}\} \subset \{\text{flabby}\}$ 3.24	H^i	$R\Gamma : D^+(Qcoh(X)) \to D^+(Vec(k))$	(Gro. vanish) 3.20 $D^b(Qcoh(X)) \to D^b(Vec(k))$		X: proper/k (Serre's thm.) 3.21 $D^b(X) \to D^b(Vec_f(k))$
$X \xrightarrow{f} Y$	$f_* : Qcoh(X) \to Qcoh(Y)$	$\{\text{injective}\} \subset \{\text{flabby}\}$ 3.24	$R^i f_*$	$Rf_* : D^+(Qcoh(X)) \to D^+(Qcoh(Y))$	(Gro. vanish) 3.22 $D^b(Qcoh(X)) \to D^b(Qcoh(Y))$		f: proper (Serre's thm.) 3.23 $D^b(X) \to D^b(Y)$
$\mathcal{F} \in Qcoh(X)$	$\mathcal{H}om(\mathcal{F},-) : Qcoh(X) \to Qcoh(X)$	$\{\text{injective}\}$	$\mathcal{E}xt^i(\mathcal{F},-)$	$R\mathcal{H}om(\mathcal{F},-) : D^+(Qcoh(X)) \to D^+(Qcoh(X))$		$\mathcal{F}$: coh. ($\mathcal{E}xt^i$(coh,coh)はcoh) $D^+(X) \to D^+(Y)$ 引き継ぐ $\Downarrow$ 2.68	X: regular ($cd(coh(X)) < \infty$) 3.26 $D^b(X) \to D^b(Y)$
$\mathcal{F}^\bullet \in D^-(Qcoh(X))$	$\mathcal{H}om(\mathcal{F}^\bullet,-) : K^+(Qcoh(X)) \to K^+(Qcoh(X))$	$\{\text{injective}^\bullet\}$ 3.25	$\mathcal{E}xt^i(\mathcal{F}^\bullet,-)$	$R\mathcal{H}om(\mathcal{F}^\bullet,-) : D^+(Qcoh(X)) \to D^+(Qcoh(X))$		$\mathcal{F}^\bullet \in D^-(X)$ $D^+(X) \to D^+(X)$	X: regular $\mathcal{F}^\bullet \in D^b(X)$ $D^b(X) \to D^b(X)$
	$\mathcal{H}om(-,-) : K^-(Qcoh(X))^{op} \times K^+(Qcoh(X)) \to K^+(Qcoh(X))$	$\{\text{injective}^\bullet\}$ 3.25	$\mathcal{E}xt^i(-,-)$	$R\mathcal{H}om(-,-) : D^-(Qcoh(X))^{op} \times D^+(Qcoh(X)) \to D^+(Qcoh(X))$		$D^-(X)^{op} \times D^+(X) \to D^+(X)$	X: regular $D^b(X)^{op} \times D^b(X) \to D^b(X)$
X: proj./k $\mathcal{F} \in Coh(X)$	$\mathcal{F} \otimes - : Coh(X) \to Coh(X)$	$\{\text{loc.free}\}$ 地の文	$\mathcal{T}or_{-i}(\mathcal{F},-)$			$\mathcal{F} \overset{L}{\otimes} - : D^-(X) \to D^-(X)$ 引き継ぐ $\Downarrow$ 2.68	X: smooth ($cd(coh(X)) < \infty$) 3.26 $D^b(X) \to D^b(X)$
X: proj./k $\mathcal{F}^\bullet \in D^-(X)$	$\mathcal{F}^\bullet \otimes - : K^-(Coh(X)) \to K^-(Coh(X))$	$\{\text{loc.free}^\bullet\}$ spec.seq.	$\mathcal{T}or_{-i}(\mathcal{F}^\bullet,-)$			$\mathcal{F}^\bullet \overset{L}{\otimes} - : D^-(X) \to D^-(X)$	X: smooth $\mathcal{F}^\bullet \in D^b(X)$ $D^b(X) \to D^b(X)$
	$- \otimes - : K^-(Coh(X)) \times K^-(Coh(X)) \to K^-(Coh(X))$	$\{\text{loc.free}^\bullet\}$ spec.seq.	$\mathcal{T}or_{-i}(-,-)$			$- \overset{L}{\otimes} - : D^-(X) \times D^-(X) \to D^-(X)$	X: smooth $D^b(X) \times D^b(X) \to D^b(X)$

[trace map $\mathrm{tr}_{\mathcal{F}^\cdot}$]

$$\left[\begin{array}{l} \mathcal{E} : \text{loc. free}. \\ \mathrm{tr}_{\mathcal{E}} : \mathcal{H}om(\mathcal{E},\mathcal{E}) \cong \mathcal{E}^\vee \otimes \mathcal{E} \xrightarrow{ev} \mathcal{O}_X. \\ \left(\mathrm{tr}_{\mathcal{E}} \in \mathrm{Hom}_{\mathrm{Coh}(X)}(\mathcal{H}om(\mathcal{E},\mathcal{E}), \mathcal{O}_X) \cong \mathrm{Hom}_{\mathrm{Coh}(X)}(\mathcal{E},\mathcal{E}) \ni \mathrm{id}_{\mathcal{E}} \right) \end{array}\right]$$

$(X : \text{reg.})$ $\mathcal{F}^\bullet \in D^b(X)$.

$$\mathrm{tr}_{\mathcal{F}^\cdot} : R\mathcal{H}om(\mathcal{F}^\cdot, \mathcal{F}^\cdot) \longrightarrow \mathcal{O}_X$$

$$\begin{array}{ccc} \mathrm{tr}_{\mathcal{F}^\cdot} \in & \mathrm{Hom}_{D^b(X)}\big(R\mathcal{H}om(\mathcal{F}^\cdot, \mathcal{F}^\cdot), \mathcal{O}_X[0]\big) \\ \shortparallel & \wr\!\shortparallel \\ \mathrm{tr}_{\mathcal{E}^\cdot} \in & \mathrm{Hom}_{D^b(X)}\big(\mathcal{H}om^\bullet(\mathcal{E}^\cdot, \mathcal{E}^\cdot), \mathcal{O}_X[0]\big) \end{array}$$

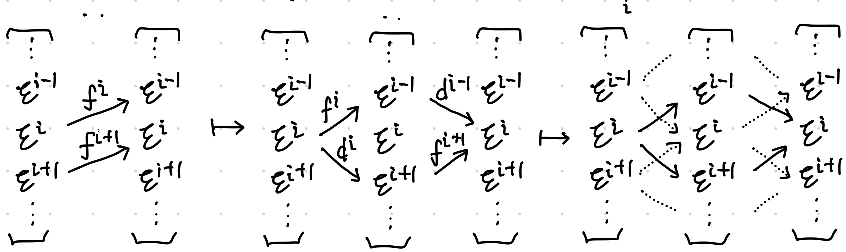
$\mathcal{E}^\bullet \xrightarrow{qis} \mathcal{F}^\bullet$: fin. loc. free. resol.

$$\begin{array}{ccc} \boxed{\mathcal{H}om^{-1} \xrightarrow{d} \mathcal{H}om^{0}} & = \mathcal{H}om(\mathcal{E}^\cdot, \mathcal{E}^\cdot) \\ \downarrow \quad \circlearrowright \quad \downarrow \mathrm{tr} & \\ 0 \longrightarrow \boxed{\mathcal{O}_X} & \end{array}$$

$$\begin{array}{c} \mathrm{Hom}_{\mathrm{Coh}(X)}\big(\mathcal{H}om^0(\mathcal{E}^\cdot, \mathcal{E}^\cdot), \mathcal{O}_X\big) \\ \wr\!\shortparallel \\ \mathrm{Hom}_{\mathrm{Coh}(X)}\big(\bigoplus_i \mathcal{H}om(\mathcal{E}^i, \mathcal{E}^i), \mathcal{O}_X\big) \\ \wr\!\shortparallel \\ \bigoplus_i \mathrm{Hom}_{\mathrm{Coh}(X)}\big(\mathcal{H}om(\mathcal{E}^i, \mathcal{E}^i), \mathcal{O}_X\big) \ni \bigoplus_i (-1)^i \mathrm{tr}_{\mathcal{E}^i} \end{array}$$

(Ex. 3.28)

$$\begin{array}{ccccc} \mathcal{H}om^{-1}(\mathcal{E}^\cdot, \mathcal{E}^\cdot) & \xrightarrow{d} & \mathcal{H}om^{0}(\mathcal{E}^\cdot, \mathcal{E}^\cdot) & \xrightarrow{\mathrm{tr}} & \mathcal{O}_X \\ \shortparallel & & \shortparallel & & \\ \bigoplus_i \mathcal{H}om(\mathcal{E}^i, \mathcal{E}^{i-1}) & \longrightarrow & \bigoplus_i \mathcal{H}om(\mathcal{E}^i, \mathcal{E}^{i}) & \xrightarrow{\sum (-1)^i \mathrm{tr}_{\mathcal{E}^i}} & \mathcal{O}_X \\ \Psi & & \Psi & & \\ (f^i) & \longmapsto & (d_{\mathcal{E}}^{i-1} \circ f^i + f^{i+1} \circ d_{\mathcal{E}}^i) & \longmapsto & \sum_i (-1)^i \mathrm{tr}_{\mathcal{E}^i}(d_{\mathcal{E}}^{i-1} \circ f^i + f^{i+1} \circ d_{\mathcal{E}}^i) = 0 \end{array}$$



$\mathrm{tr}(\searrow\nearrow) = \mathrm{tr}(\nearrow\searrow)$ より隣接項が
うちけす.

[双対 $(-)^\vee$]

$$\left[\begin{array}{l} \mathcal{E} : \text{loc. free} \\ \mathcal{E}^\vee = \mathcal{H}om(\mathcal{E}, \mathcal{O}_X) : \text{loc. free} \end{array}\right]$$

$$\mathcal{F}^\bullet \in D^-(Qcoh(X))$$

$$\mathcal{F}^{\bullet\vee} := R\mathcal{H}om(\mathcal{F}^\bullet, \mathcal{O}_X) \in D^+(Qcoh(X))$$

$\mathcal{F}^\bullet$: loc. free からなる複体なら

$$\mathcal{F}^{\bullet\vee} = \left[\cdots \to \mathcal{F}^{i+1\vee} \to \mathcal{F}^{i\vee} \to \mathcal{F}^{i-1\vee} \to \cdots\right]. \quad \cdots \star$$

$(X : \text{reg.})$ $\mathcal{F}^\bullet \in D^b(X)$ に対し $\mathcal{F}^{\bullet\vee} \in D^b(X)$ となる.

[逆像 f^*]

$$Y \xleftarrow{\quad\quad\quad} X \quad : f$$

$$\begin{tikzcd}
f^* \arrow[d, equal] & : & Sh_{\mathcal{O}_Y}(Y) \arrow[rr] \arrow[d, equal] & & Sh_{\mathcal{O}_X}(X) \arrow[d, equal] \\
\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}- & : & Sh_{\mathcal{O}_Y}(Y) \arrow[r, "f^{-1}", "\text{exact}"'] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Sh_{f^{-1}\mathcal{O}_Y}(X) \arrow[r, "\mathcal{O}_X\otimes -", "\text{r.exact}"'] \arrow[d] \arrow[dr, phantom, "\Downarrow"] & Sh_{\mathcal{O}_X}(X) \\
\mathcal{O}_X \overset{L}{\otimes}_{f^{-1}\mathcal{O}_Y} f^{-1}- & : & D^-(Sh_{\mathcal{O}_Y}(Y)) \arrow[r, "f^{-1}"] & D^-(Sh_{f^{-1}\mathcal{O}_Y}(X)) \arrow[r, "\mathcal{O}_X\overset{L}{\otimes} -"] & D^-(Sh_{\mathcal{O}_X}(X)) \\
Lf^* & : & D^-(Y) \arrow[u, hook] \arrow[rr] & & D^-(X) \arrow[u, hook]
\end{tikzcd}$$

Lem. 3.19. $i : T \hookrightarrow X$: cl.sbsch. $\mathcal{F} \in D^b(X)$

$$\mathrm{Supp}(\mathcal{F}) \cap T = \mathrm{Supp}(Li^*\mathcal{F})$$

prf. (⊇)

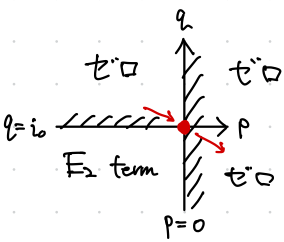
$$\begin{tikzcd} U \cap T \arrow[r, hook] \arrow[d, "i_U"] \arrow[dr, phantom, "\circlearrowleft"] & T \arrow[d, "i"] \\ U := \mathrm{Supp}(\mathcal{F})^c \arrow[r, hook] & X \end{tikzcd} \qquad \begin{tikzcd} 0 = (Li^*\mathcal{F})|_{U\cap T} \arrow[dr, phantom, "\circlearrowleft"] & Li^*\mathcal{F} \arrow[l, mapsto] \\ 0 = \mathcal{F}|_U \arrow[u, "Li_U^*"] & \mathcal{F} \arrow[l, mapsto] \arrow[u, "Li^*"'] \end{tikzcd}$$

(⊆) $x \in \mathrm{Supp}\,\mathcal{F}\left(= \bigcup \mathrm{Supp}\,\mathcal{H}^i(\mathcal{F})\right) \cap T$ を取る.

$x \in \mathrm{Supp}\,\mathcal{H}^{i_0}(\mathcal{F})$ なる最大の i_0 を取る.

i.e. $\mathcal{H}^{i_0}(\mathcal{F}) \otimes k(x) \neq 0$

$$E_2^{p,q} = \mathcal{T}or_{-p}(\mathcal{H}^q(\mathcal{F}), k(x)) \Rightarrow E^{p+q} = \mathcal{T}or_{-(p+q)}(\mathcal{F}, k(x)) = \mathcal{H}^{p+q}(\mathcal{F} \otimes k(x))$$



より $0 \neq E_2^{0,i_0} = E_\infty^{p,q} = E^{p+q} = \mathcal{H}^{p+q}(\mathcal{F} \otimes k(x))$

|| 構成より

$L[\{x\} \hookrightarrow T]^* \circ Li^*\mathcal{F} = L[\{x\} \hookrightarrow X]^*\mathcal{F}$.

(Ex.3.30) より $x \in \mathrm{Supp}(Li^*\mathcal{F})$ □

(Ex. 3.30 : $x \in \mathrm{Supp}\,\mathcal{F} \Leftrightarrow L[\{x\} \hookrightarrow X]^*\mathcal{F} \neq 0$)

⇒ : まさにこの Ex. を使う直前までで示したこと.

⇐ : 自明 (⊇パートで示したこと)

Lem. 3.31

$$\begin{array}{ccc} S_x & \xhookrightarrow{i_x} & S \\ \downarrow & \lrcorner & \downarrow \\ \{x\} & \hookrightarrow & X \end{array}$$

$$\begin{array}{ccccc} & \mathrm{Coh}(S_x) & & \mathrm{Coh}(S) & \\ & \downarrow & & \downarrow & \\ \text{if } \forall x,\ \exists \bigcirc \in & D^b(S_x) & \xleftarrow{Li_x^*} & D^b(S) & \ni \exists \bigcirc : \mathrm{flat}/X \text{ then} \\ & \rotatebox{-90}{$\in$}\ Li_x^*Q & \longleftarrow\!\!\shortmid & Q & \end{array}$$

i.e. $\displaystyle\bigcap_{x\in X} (Li_x^*)^{-1}(\mathrm{Coh}(S_x)) \subseteq \mathrm{Coh}_{\mathrm{flat}/X}(S)$

$$Q = [\cdots \to Q^{m-1} \to Q^m \to 0 \to \cdots]$$

wts : ① $m=0$, ② $\mathcal{H}^0(Q)$: flat/X, ③ $\mathcal{H}^{q<0}(Q)=0$

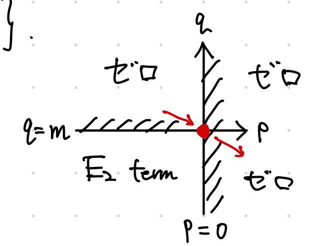
prf.

$$E_2^{p,q} = \mathcal{H}^p(Li_x^*\mathcal{H}^q(Q)) \Rightarrow E^{p+q} = \underbrace{\mathcal{H}^{p+q}(Li_x^*Q)}$$

$=0$ when $p+q\neq 0$ ($\because \dim\{x\}=0$)

① $m := \max\{q \mid \mathcal{H}^q(Q)\neq 0\}$.

$\exists x,\ E_2^{0,m} = \mathcal{H}^0(Li_x^*\mathcal{H}^m(Q)) \neq 0$



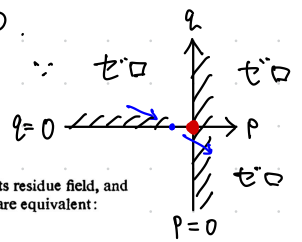
より $0\neq E_2^{0,m} = E_\infty^{0,m} = E^m$

$\rightsquigarrow m=0$

② $E_2^{-1,0} = \mathcal{H}^{-1}(Li_x^*\mathcal{H}^0(Q)) = 0$.

$\quad\quad \mathrm{Tor}_1(k(x), \mathcal{H}^0(Q))$

$\because$



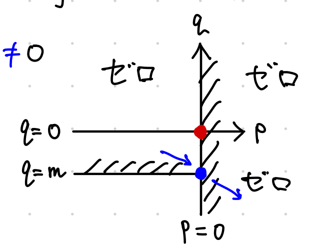
「
15. Let A be a Noetherian local ring, $\mathfrak{m}$ its maximal ideal and k its residue field, and let M be a finitely generated A-module. Then the following are equivalent:

i) M is free;
ii) M is flat;
iii) the mapping of $\mathfrak{m}\otimes M$ into $A\otimes M$ is injective;
iv) $\mathrm{Tor}_1^A(k, M) = 0$.
」

$\longrightarrow \mathcal{H}^0(Q)$: flat/X.

③ $m := \max\{q<0 \mid \mathcal{H}^q(Q)\neq 0\}$ の存在を仮定する.

$\exists x,\ E_2^{0,m} = \mathcal{H}^0(Li_x^*\mathcal{H}^m(Q)) \neq 0$



$E^m\neq 0$ となり矛盾.

$\square$

☆ Compatibilities （全てのスキームは sm, proj. /k）

$$
\begin{tikzcd}
X \arrow[d, "f"] & D^b(X) \times D^b(X) \arrow[r, "\otimes^L"] & D^b(X) \arrow[d, "Rf_*"] \\
Y & D^b(Y) \times D^b(Y) \arrow[r, "\otimes^L"] & D^b(Y)
\end{tikzcd}
$$

(projection formula)

$$
\begin{tikzcd}
X \arrow[d, "f"] & D^b(X) \times D^b(X) \arrow[r, "\otimes^L"] & D^b(X) \\
Y & D^b(Y) \times D^b(Y) \arrow[u, "Lf^* \times Lf^*"] \arrow[r, "\otimes^L"] & D^b(Y) \arrow[u, "Lf^*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[d, "f", "\mathrm{proj.}"'] & D^b(X)^{op} \times D^b(X) \arrow[dr, "\mathrm{Hom}"] & \\
 & & \mathrm{Vec}(k) \\
Y & D^b(Y)^{op} \times D^b(Y) \arrow[ur, "\mathrm{Hom}"'] &
\end{tikzcd}
$$

$(Lf^* \dashv Rf_*)$

$$
\begin{tikzcd}
D^b(X) \times D^b(X) \arrow[dr, "\otimes^L"] & \\
D^b(X)^{op} \times D^b(X) \times D^b(X) \arrow[u, "R\mathcal{H}om"] \arrow[d, "\otimes^L"'] & D^b(X) \\
D^b(X)^{op} \times D^b(X) \arrow[ur, "R\mathcal{H}om"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D^b(X) \times D^b(X) \arrow[dr, "R\mathcal{H}om"] & \\
D^b(X)^{op} \times D^b(X)^{op} \times D^b(X) \arrow[u, "\otimes^L"] \arrow[d, "R\mathcal{H}om"'] & D^b(X) \\
D^b(X)^{op} \times D^b(X) \arrow[ur, "R\mathcal{H}om"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D^b(X)^{op} \times D^b(X) \arrow[dr, "R\mathcal{H}om"] & \\
D^b(X) \times D^b(X)^{op} \times D^b(X) \arrow[u, "R\mathcal{H}om(2{,}1)"] \arrow[d, "\otimes^L"'] & D^b(X) \\
D^b(X) \times D^b(X) \arrow[ur, "R\mathcal{H}om"'] &
\end{tikzcd}
$$

$$
\begin{tikzcd}
D^b(X) \times D^b(X) \arrow[dr, "\overset{L}{\otimes}"] & \\
D^b(X)^{op} \times D^b(X) \arrow[u, "(-)^\vee \times \mathrm{id}"] \arrow[r, "R\mathcal{H}om"'] & D^b(X)
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & D^b(\mathrm{Vec}_f(k)) \\
D^b(X)^{op} \times D^b(X) \arrow[ur, "R\mathrm{Hom}"] \arrow[dr, "R\mathcal{H}om"'] & \\
 & D^b(X) \arrow[uu, "R\Gamma"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \arrow[d, "f"'] \\
Y
\end{tikzcd}
\qquad
\begin{tikzcd}
D^b(X)^{op} \times D^b(X) \arrow[r, "R\mathcal{H}om"] & D^b(X) \\
D^b(Y)^{op} \times D^b(Y) \arrow[u, "Lf^* \times Lf^*"] \arrow[r, "R\mathcal{H}om"] & D^b(Y) \arrow[u, "Lf^*"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
X \times_Z Y \arrow[r, "v"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner"] & X \arrow[d, "f"] \\
Y \arrow[r, "u"', "\text{flat}"] & Z
\end{tikzcd}
\qquad
\begin{tikzcd}
D^b(X \times_Z Y) \arrow[d, "Rg_*"'] & D^b(X) \arrow[l, "v^*"'] \arrow[d, "Rf_*"] \\
D^b(Y) & D^b(Z) \arrow[l, "u^*"]
\end{tikzcd}
\quad \text{(flat base change)}
$$

(次節)

$$
\begin{tikzcd}
D^b(X)^{op} \times D^b(X) \arrow[r] \arrow[dd, "Rf_*"'] & D^b(X) \arrow[dd, "Rf_*"] \\
 & \\
D^b(Y)^{op} \times D^b(Y) \arrow[r, "R\mathcal{H}om"'] & D^b(Y)
\end{tikzcd}
$$

$$
D^b(X) \xleftarrow{S_X} D^b(X) \xleftarrow{Lf^*} D^b(Y) \xrightarrow{S_Y} D^b(Y)
$$

$$
R\mathcal{H}om: D^b(X)^{op} \times D^b(X) \to D^b(X)
$$

$S_X = - \otimes \omega_X[\dim X]$

(Grothendieck-Verdier duality)

4. Grothendieck-Verdier duality

Thm. 3.34 (Grothendieck-Verdier duality) X, Y : sm.

$f : X \to Y$, $\mathcal{F} \in D^b(X)$, $\mathcal{E} \in D^b(Y)$ $\quad \dim f := \dim X - \dim Y$

$$\rightsquigarrow Rf_* R\mathcal{H}om(\mathcal{F}, \underbrace{Lf^*(\mathcal{E} \otimes \omega_Y^*) \otimes \omega_X[\dim f]}_{\text{ここを } f^!\mathcal{E} \text{ とまとめると}}) \cong R\mathcal{H}om(Rf_*\mathcal{F}, \mathcal{E})$$

Cor. 3.35 $\quad Rf_* \dashv f^! = Lf^*(- \otimes \omega_Y^*) \otimes \omega_X[\dim f]$

$\omega_f = f^*\omega_Y^* \otimes \omega_X$ とかけば $= - \otimes \omega_f[\dim f]$

$$\begin{tikzcd} D^b(X) \\ D^b(Y) \arrow[u, "Lf^*"] \end{tikzcd} \dashv \begin{tikzcd} D^b(X) \arrow[d, "Rf_*"] \\ D^b(Y) \end{tikzcd} \dashv \; f^! := \left\{ \begin{tikzcd} D^b(X) \arrow[rd, leftrightarrow, "\mathbb{D}_X"] & \\ D^b(X) \arrow[u, "S_X"] & D^b(X)^{op} \arrow[l, "(-)^\vee"'] \\ D^b(Y) \arrow[u, "Lf^*"] & D^b(Y)^{op} \arrow[l, "(-)^\vee"] \arrow[u, "Lf^*"] \\ D^b(Y) \arrow[u, leftarrow, "S_Y"] \arrow[ru, leftrightarrow, "\mathbb{D}_Y"'] & \end{tikzcd} \right.$$

$\mathbb{D}_X = R\mathcal{H}om(-, \omega_X[\dim X])$

$\mathbb{D}_X^2 = \mathrm{id}.$

dualizing functor.

exc. 3.36 $\quad Rf_* \circ \mathbb{D}_X \simeq \mathbb{D}_Y \circ Rf_* \quad (\Leftrightarrow \text{Thm.})$

Rmk. 3.37

1) smoothでなくても, $\mathbb{D}_X = R\mathcal{H}om(-, \text{dualizing complex})$ はある.

CM $\quad \Leftrightarrow \quad$: coh. shf.

Gor. $\quad \Leftrightarrow \quad$: inv. shf.

2) $f : X \to \operatorname{Spec} k$ で Serre duality を復元する.

3) $Rf_* \dashv f^! = S_X \circ Lf^* \circ S_Y$ は, Serre functorの一般論から既知.

Cor. 3.38 exc. 3.39 $i : X \hookrightarrow Y$: cl.imm. of sm.sch. codim $= c$

$$\mathrm{Hom}_X(\mathcal{F}, \underbrace{Li^*(\mathcal{E}) \otimes \omega_i[-c]}) \cong \mathrm{Hom}_Y(i_*\mathcal{F}, \mathcal{E})$$ を直接示せ.

ans.

$$\overbrace{i^{-1}i_*(Li^*(\mathcal{E})\otimes\omega_i[-c])}^{\wr\| \ i^{-1}i_* = \mathrm{id}} \qquad \mathrm{Hom}_X(\mathcal{F}, i^{-1}R\mathcal{H}om_Y(i_*\mathcal{O}_X, \mathcal{E}))$$

$\wr\|$ [Ha, III. Ex. 6.10] + cl.imm. は fin.

$\wr\|$ proj. formula $\qquad\qquad \wr\|$

$$\mathcal{E} \otimes^L \underbrace{i_*\omega_i[-c]} \quad \overset{?}{=} \quad \underbrace{(i_*\mathcal{O}_X)}^{\vee} \otimes^L \mathcal{E}$$

- X : sm. より, $\mathcal{I}_X$ は局所的に (e.g. U 上) 正則列 $x_1, \dots, x_c$ で生成される.
  すると $i_*\mathcal{O}_X|_U = (\mathcal{O}_Y/\mathcal{I}_X)|_U$ の Koszul 複体による局所自由分解

$$\Big[\underset{\det E_U}{\overset{\|}{\bigwedge^c E_U}} \to \cdots \to \bigwedge^2 E_U \to \underset{\bigoplus_{i=1}^c \mathcal{O}_U e_i}{\overset{\|}{E_U}} \xrightarrow{e_i \mapsto x_i} \overset{[0]}{\mathcal{O}_U}\Big] \xrightarrow{qis} i_*\mathcal{O}_X|_U \qquad \text{: free resol.}$$

がある.

- loc. free cplx に対しては双対は pointwise 双対 + 逆順 となっていて (★参照)

$$(i_*\mathcal{O}_X|_U)^\vee \xleftarrow{qis} \Big[\overset{[0]}{\mathcal{O}_U^\vee} \to E_U^\vee \to \bigwedge^2 E_U^\vee \to \cdots \to \bigwedge^c E_U^\vee\Big]$$

- $\mathrm{rk}\, E_U = c$ で $\bigwedge^p E_U^\vee \otimes \bigwedge^{c-p} E_U^\vee \to \bigwedge^c E_U^\vee$ は perfect pairing [Ha, II, Ex. 5.16(b)] であり,

$$\cong \begin{bmatrix} \bigwedge^c E_U & & \bigwedge^{c-1} E_U & & \bigwedge^{c-2} E_U & & & & \overset{[c]}{\mathcal{O}_U} \\ \otimes & \to & \otimes & \to & \otimes & \to & \cdots & \to & \otimes \\ \det E_U^\vee & & \det E_U^\vee & & \det E_U^\vee & & & & \det E_U^\vee \end{bmatrix} \text{: もとの Koszul 複体の } c \text{ シフト}$$

(実は Cor. 3.40) コホモロジーはここのみ.

$$= i_*\mathcal{O}_X|_U \otimes \det E_U^\vee[-c] \quad \text{この形は貼り合う.}$$

- いま X : sm. より, 自然に $\mathcal{O}_U/\mathcal{I}_X|_U \otimes E_U \cong \mathcal{I}/\mathcal{I}^2|_U$ (local algebra)
  $1 \otimes e_i \mapsto \overline{x_i}$
- 両辺の台 $\subseteq X$ より i^{-1} で $X \cap U$ 上の層と思う.
- $\mathcal{H}om_{X\cap U}(-, \mathcal{O}_{X\cap U})$ で $\qquad i_*\mathcal{O}_X \otimes E_U^\vee \cong (\ \text{〃}\ )^\vee \overset{\mathrm{def}}{\cong} \mathcal{N}_{X/Y}|_U$
- det をとると $\qquad i_*\mathcal{O}_X \otimes \det E_U^\vee \cong \det \mathcal{N}_{X/Y}|_U \cong i_*\omega_i$
  [Ha, II. 8.20] □